A qualitative Langevin-like model for the coexistence of two distinct granular temperatures

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Abstract

In the present work, we study qualitatively the physics of granular temperature coexistence, for a mixture of two different species. Our model captures its essential aspects and this allows us to get insights on the physical mechanisms of distinct temperature coexistence, in a way which is not obscured by the complexities of kinetic theories or numerical simulations. Our simple model is consistent with limit situations where we should expect equality for the granular temperatures for the mixture.

1 Introduction

Granular Systems (GS) are ubiquitous in Nature [1]. They consist of large numbers of distinct, inelastic, rough grains. When studying GS, we are faced with mesoscopic collective effects, such as frictional forces, that are not present in molecular gases [1–3]. The presence of energy dissipation during internal collisions [4] make GS very different from molecular gases: true equilibrium is only possible when the total kinetic energy is completely dissipated. Thus, in order to keep a GS in a steady-state, we need to inject energy through the boundaries or by means of a coupled heath bath [5–8]. The parameter used to account for the kinetic energy present in a GS is the so called granular

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temperature, i.e., the average kinetic energy per grain. It is clearly related to the usual microscopic definition of temperature in statistical mechanics [9].

Thus, we define macroscopic granular quantities taking inspiration from thermodynamic concepts familiar to us. For instance, pressure and granular number density, among others, are widely used when studying GS [9,10]. However, the naive application of thermodynamic concepts to GS can sometimes be misleading. Take pressure for instance: for molecular gases, as we increase the number density, at fixed energy feeding rate, the pressure should increase. However, for GS, the reverse happens for high enough densities [7]. This tells us to be careful when applying thermodynamic concepts, such as the granular temperature T_g, to GS.

Thermodynamics is based on a few simple principles, much in the same way as Euclidean geometry is based on five postulates. However, we can build non-Euclidean geometries from the first four postulates and the negation of the fifth. We ask ourselves: is there a similar hope for GS thermodynamics? In this spirit, we start by studying the most basic law of granular thermodynamics: the zeroth law, or the temperature equilibrium law. Our model is a very simple one, based on a Langevin-like approach to the collisions of different granular species in a mixture.

We study the transient behavior of a binary granular gas mixture evolving toward a steady-state [11]. The mixture components are smooth, inelastic identical spheres of two type: a species of mass M and another species of mass $m \leq M$ and same radius. A convenient Langevin-like collision term is introduced, aiming to take into account the more complicated aspects of the physics in a simplified, effective way. Experimental realizations of granular temperature coexistence [12] have shown some characteristics that can be reproduced by the present model. In our opinion, the model's main advantage is that it permits us to visualize some important physical aspects of an actual system in a mathematical setting which is much simpler then that of more complete theories [13].

This paper is organized as follows. In Section 2, we study the behavior of an intruder in a bath of particles. In Section 3, we analyze the mixture of granular species while we calculate the temperature ratio in Section 4. We conclude briefly in Section 5.

2   Test particle in a granular bath

We study the behavior of a 1D test particle of mass M, velocity V, immersed in granular bath of particles of mass $m = \gamma M$ and same radius of the test

particle [14]. The granular bath interacts with the test particle through frontal inelastic collisions characterized by a coefficient of restitution $0 < \varepsilon < 1$.

We assume that the typical bath particle has a **constant driving velocity** v_D and we suppose that collisions happen with equal probabilities for the direction of the bath particle's velocity. The choice above is a reasonable one, since there are always four possible ways for a bath particle to collide with the test particle (with velocity V): two frontal collisions (with V and $\pm v_D$) and two rear ones (again with V and $\pm v_D$). For each case, regardless of whether $|V| > v_D$ or $|V| < v_D$, a simple analysis shows that the next collision will occur either with a $+v_D$ or with a $-v_D$ bath particle. A further implicit simplifying hypothesis is that we ignore the dependence of the collision rate on the relative velocity. This is similar to the so called Maxwell model [15]. We thus write the momentum conservation and the inelastic behavior equation as

$$MV \pm mv_D = MV' + mv', \tag{1}$$
$$V' - v' = -\varepsilon(V - (\pm v_D)). \tag{2}$$

After a little manipulation, we obtain

$$V' = \left(\frac{1 - \gamma\varepsilon}{1 + \gamma}\right)V \pm v_D\left(\frac{\gamma(1 + \varepsilon)}{1 + \gamma}\right). \tag{3}$$

The test particle's change in velocity is thus

$$\Delta V = V' - V = \beta[-V + \eta v_D], \text{ where } \beta = \frac{\gamma(1 + \varepsilon)}{1 + \gamma}, \tag{4}$$

and η is a random variable with values ± 1 with equal probability distribution, $< \eta >= 0$, $\eta^2 = 1$.

The test particle's collisional history is determined by its sequence of collisions with the bath particles. We assign an independent random variable η_{N-1} for the N-th collision. Then, we can write the velocity of the test particle, after the $N-$th collision as

$$V_N = (1 - \beta)^N V_0 + \beta v_D \sum_{i=0}^{N-1}(1 - \beta)^{N-1-i}\eta_i, \tag{5}$$

where V_0 is the initial velocity of the test particle and

$$< \eta_i >= 0, \quad < \eta_i\eta_j >= \delta_{i,j}, \quad \eta_i^2 = 1. \tag{6}$$

3

It can be seen from above that the memory of the initial velocity is lost exponentially fast.

Next, we calculate the cases of the test particle being a bath particle ($\gamma = 1$) and that of it being a massive intruder ($\gamma \neq 1$) [14,16]. In these two limiting cases, the quadratic velocities will be in equilibrium with the bath and their ratio should reflect the ratio between test particle-bath particle quadratic ratio. Obviously, we have $v_{bath} \neq v_D$ since we allow v_{bath} to fluctuate while supposing v_D fixed. Thus, it is straightforward to show that

$$v_{bath}^2 = \lim_{N \to \infty, \gamma = 1} <V_N^2> = v_D^2 \frac{1 + \varepsilon}{3 - \varepsilon}, \tag{7}$$

$$v_{test}^2 = \lim_{N \to \infty} <V_N^2> = v_D^2 \frac{\gamma(1 + \varepsilon)}{2 + \gamma(1 - \varepsilon)}. \tag{8}$$

From Eq. 7-8, we obtain the granular temperature ratio in the steady state for a test particle immersed in a granular bath:

$$\frac{T_g^{bath}}{T_g^{test}} = \frac{\frac{1}{2}m <v_{bath}^2>}{\frac{1}{2}M <v_{test}^2>} = \frac{\frac{m}{2}v_D^2 \frac{1+\varepsilon}{3-\varepsilon}}{\frac{M}{2}v_D^2 \frac{\gamma(1+\varepsilon)}{2+\gamma(1-\varepsilon)}} = \frac{2 + \gamma(1 - \varepsilon)}{3 - \varepsilon}. \tag{9}$$

In order to compare the result of Eq. 9 with the literature on the tracer limit, we see that in the quasi-elastic case, $\varepsilon \to 1$, we have up to first order in the limit of the heavy tracer $\gamma \to 0$

$$\frac{T_g^{test}}{T_g^{bath}} \approx 1 + \frac{1 - \varepsilon}{2}. \tag{10}$$

From reference [13], which agrees with references [14,16], we observe that from Eq. 31 and Fig. 3, the same limit above is obtained. The present model seems to have better agreement with kinetic theories on this region. The case of larger γ does not agree so well. However, its aim is not to obtain accurate quantitative results, but to get insights into the mechanism of granular temperature coexistence.

The cases of: same species ($\gamma = 1$) and elastic motion ($\varepsilon = 1$) imply that $T_g^{bath} = T_g^{test}$, as expected.

We observe that the above difference in granular temperatures is due to the factor β being smaller for the massive intruder, $\gamma < 1$ [14], leading to a smaller loss of speed during an inelastic collision giving

$$(1 - \beta_{int}) = \left(1 - \frac{2\gamma}{1 + \gamma} \frac{(1 + \varepsilon)}{2}\right) > (1 - \beta_{bath}) = \left(1 - \frac{(1 + \varepsilon)}{2}\right),$$

(see Eq. 5). This reflects the fact that, due to its inertia, a massive inelastic intruder will lose comparatively less energy to a lighter bath than a less massive particle would.

Our next step will be to characterize the mixture of different species.

3 Granular mixture

We suppose that the granular fluid under study is composed of homogeneous particles of identical diameters. Similarly to the previous section, any chosen particle's collisions will occur with a bath of particles. But now this bath consists of a mix of two types of particles with different constant driving velocities, namely v_D for particles m and V_D for particles M [11] (the above mentioned particle belongs to one of these families). The coefficients of restitution are ε_1, ε_2 and ε' corresponding to the mm, MM, mM (or Mm) collision possibilities. In what follows, we use the definitions

$$\alpha_1 = \frac{1 + \varepsilon_1}{2}, \ \alpha_2 = \frac{1 + \varepsilon_2}{2}, \ \beta = \frac{\gamma(1 + \varepsilon')}{1 + \gamma} = \gamma\zeta. \tag{11}$$

We need two families of random variables in order to describe the chosen particle's $N-$th collision of a: a variable $\theta_{m,N}$ or $\theta_{M,N}$, to describe with what species the collision occurs; another, $\eta_{m,N}$ or $\eta_{M,N}$, to give the direction of the bath particle. These variables have the following distributions: $\theta_{m,N} = 1$ with probability x, $\theta_{m,N} = 0$ with probability $1 - x$, $\theta_{M,N} = 1$ with probability $1 - x$, $\theta_{M,N} = 0$ with probability x, where

$$x = \frac{\rho_m}{\rho_m + \rho_M}, \tag{12}$$

and ρ_m and ρ_M are the species number densities; the $\eta-$variables are distributed with equal probabilities.

According to the distributions above and skipping the algebra, in the limit $N \to \infty$ we obtain: $< v_\infty > = 0$ and

$$< v_\infty^2 > = \frac{x\alpha_1^2 v_D^2 + (1 - x)\zeta^2 V_D^2}{1 - (x(1 - \alpha_1)^2 + (1 - x)(1 - \zeta)^2)}. \tag{13}$$

Similarly, we have $< V_\infty > = 0$ and

$$< V_\infty^2 > = \frac{(1-x)\alpha_2^2 V_D^2 + x\beta^2 v_D^2}{1 - ((1-x)(1-\alpha_2)^2 + x(1-\beta)^2)}. \tag{14}$$

4 Temperature ratio: general situation

As in the case of a test particle, we define the granular temperature ratio as

$$\frac{T_g^m}{T_g^M} = \frac{\frac{1}{2}m < v_\infty^2 >}{\frac{1}{2}M < V_\infty^2 >} = \gamma \frac{< v_\infty^2 >}{< V_\infty^2 >} = \gamma \frac{v_D^2}{V_D^2}, \tag{15}$$

where we assumed that the same relationship holds both for the steady-state velocities and the driving velocities.

Replacing Eq. 13 and Eq. 14 into Eq. 15, and solving for $\frac{T_g^m}{T_g^M}$, yields

$$\frac{T_g^m}{T_g^M} = \gamma \left\{ \frac{[x\alpha_1^2 - (1-x)\alpha_2^2 A]}{2x\beta^2 A} \right.$$
$$\left. + \frac{\sqrt{[x\alpha_1^2 - (1-x)\alpha_2^2 A]^2 + 4x(1-x)\beta^2\zeta^2 A}}{2x\beta^2 A} \right\}, \tag{16}$$

where

$$A = \frac{1 - x(1-\alpha_1)^2 - (1-x)(1-\zeta)^2}{1 - (1-x)(1-\alpha_2)^2 - x(1-\beta)^2}. \tag{17}$$

We notice, in Eq. 16, that the dependence of $\frac{T_g^m}{T_g^M}$ on γ is stronger than on the inelasticity parameters, as mentioned in reference [12]. Some important physical limits are listed below.

Elastic limit. In this limit, $\varepsilon_1 = \varepsilon_2 = \varepsilon' = 1$, and we have

$$\alpha_1 = \alpha_2 = 1, \ \beta = \gamma\zeta = \frac{2\gamma}{1+\gamma} \Rightarrow \frac{T_g^m}{T_g^M} = 1, \tag{18}$$

for any value of γ. The elastic equilibrium regime is such that all granular temperatures are the same, as expected.

Same species, same inelasticity. In this case we set $\gamma = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon' \equiv \varepsilon$. We have now

$$\alpha_1 = \alpha_2 = \beta = \zeta = \frac{1+\varepsilon}{2}.$$

It is also trivial to check that $\frac{T_g^m}{T_g^M} = 1$. But, for $\gamma \neq 1$ and $\varepsilon_1, \varepsilon_2, \varepsilon' < 1$ the granular temperature ratio is non trivial, assuming values different from 1.

Tracer limit. It is straightforward to show that we recover Eq. 9 when $x \rightarrow 0$ [14,16]:

$$\lim_{x \to 0} \frac{T_g^{bath}}{T_g^{test}} = \frac{2 + \gamma(1 - \varepsilon)}{3 - \varepsilon}. \tag{19}$$

Comparison with experiments. In a recent work, Feitosa and Menon [12] used beads of equal diameters, made from different materials in order to study the granular temperature behavior of a granular mix. According to their experimental data, for a 1/2-1/2 mix of glass-brass: $x = 0.5$, $\gamma = 0.28$, $\varepsilon_1 = 0.83$, $\varepsilon_2 = 0.61$, $\varepsilon' = 0.72$, $\alpha_1 = 0.92$, $\alpha_2 = 0.81$, $\beta = 0.38$ and $\zeta = 1.34$. We assumed $\varepsilon' = \frac{\varepsilon_1 + \varepsilon_2}{2}$ in order to carry out the calculations. By looking at the data in reference [12], we estimate $\frac{T_g^m}{T_g^M} \approx 0.8$ as the plateau value for large velocities, where the above ratio seems to converge to.

From our model, and using the data above, we obtain

$$\frac{T_g^m}{T_g^M} = 0.86,$$

which is not too far from the experimental result above. But we need to keep in mind that the present model is basically qualitative. It is not sophisticated enough in order to obtain precise quantitative agreement with the experiments.

5   Conclusions

A simple Langevin-like collisional model, valid for any mass ratio and inelasticities values, which assumes a bath of grains interacting with test particles, is used to study the steady-state average kinetic energy of a mixture of distinct species of grains. The collisions are modeled by a Langevin-like impulsive term. The collisional history of the test particle is obtained and the limiting mean square velocity calculated. We compare the mean kinetic energy for each species. There results a difference in coexisting granular temperatures. We verify that this is due to inertial effects during a collision of massive and light particles.

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